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Systematic Relations Between Futures and Expected Spot Prices: The Issue of Normal Backwardation

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Systematic Relations Between Futures and Expected Prices: The Issue of Normal Backwardation

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This paper, concentrating on the empirical issue, substantially extends an earlier version, BEBR Faculty Working Paper No. 927.

Abstract

The hypothesis that futures prices are the unbiased estimates of expected spot prices has often been disguised under the term, market efficiency, in the financial analysis of futures markets. This paper, in an attempt to fill the gap between the theory and the observed facts, formulates the pricing model of the futures contracts and derives the systematic relations, which are empirically testable, between the futures prices and the expected spot prices. The derived implications of the theoretical models are significantly consistent with the empirical findings using six physical commodities and four foreign currencies with five different times to maturity each (Gold, Silver, Silver Coin, Platinum, Copper, Plywood, Swiss Franc, British Pound, German Mark, Japanese Yet). In the process, it is clearly shown that the conventional hypothesis that the normal backwardation is due to market inefficiency can be rejected.

Systematic Relations between Futures and Expected Spot Prices: The Issue of Normal Backwardation

I. Introduction

In the applied capital market theories, the expectation models of asset prices have attracted a lot of attentions theoretically and empirically. Especially, the hypothesis that futures prices are the unbiased estimates of expected spot prices has been often disguised under the term, market efficiency, in the financial analysis of futures markets.[1] Thus, the observed divergencies, if any, between the futures prices and the expected spot prices have often been attributed to the market inefficiency (Rendleman and Carabini [25], Cornell [7], Geweke and Feige [10], Burger, Lang and Rasche [5] and Poole [24]). Furthermore, the large number of papers in this area have tended to build upon each other as economic models were checked with empirical evidence and extensions or modifications were created only to be tested again: theories on the futures contracts have tended to be created to fit the observed facts, not vice versa. Coother [6], Dusak [9], Houthakker [14], Rockwell [27] and Bodie and Rosansky [3] examined the rate of returns of futures contracts, reaching different conclusions. Recently, O'Brien and Schwarz [20] observed the downward biased futures prices for the commodity, Gold. However, as Wilson [31] pointed out, O'Brien and Schwarz did not delve into the question of why the downward bias of futures prices should obtain.

This paper, in an attempt to partially fill the gap between the theory and the observed facts, formulates the pricing model of the futures contracts and derives the systematic relations, which are called

testable, between the future prices and the expected spot prices. Specifically, it will be examined under what conditions the biased futures prices, upward as well as downward, can be observed without relying upon the market inefficiency concepts. The pricing equation of futures contract is developed through the basic microeconomic theories, the market clearing condition and the first order condition for the expected utility maximization in the absence of transaction costs and in the presence of uncertainty. In turn, some causal relations between the futures and the expected spot prices are derived and tested using six basic commodities and four foreign currencies: Gold, Silver, Silver Coin, Copper, Plywood, Platinum, Swiss Franc, British Pound, German Mark, Japanese Yen. In the process, it is clearly shown that the systematic bias of the futures price, if any, is not inconsistent with market efficiency.

This paper is organized as follows. Section II briefly reviews the theory relevant to the systematic bias of the futures price against the expected spot price, the issue of "normal backwardation." The pricing equation of futures contracts and the empirical testable hypotheses are derived in Section III. Section IV describes the data and discuss the empirical results. Section V summarizes the paper briefly.

II. Normal Backwardation

The "normal backwardation" refers to the process in which the futures prices are systematically downward biased estimates of the expected spot prices over time. The opposite of this process is called the "normal contango," where the futures prices are upward biased estimates of the expected spot prices. The hypothesis, "normal backwardation," was

developed and supported mainly by Keynes [16], Hicks [12] and Houthakker [14]. The supporters for this normal backwardation hypothesis view the futures market as a semi-efficient mechanism in which risk averse speculators or hedgers insure other risk averse investors who hold positive stocks of commodities against changes in prices. A couple of sentences in "Value and Capital" of Hicks [12] summarizes the major points in this theory:

> futures prices are nearly always made partly by speculators, who seek a profit by buying futures when the futures price is below the spot price they expect to rule on the corresponding date ... a speculator will only be willing to go on buying futures so long as the futures price remains definitely below the spot price he expects: for it is the difference between these prices which he can expect to receive as a return for his risk-bearing and it will not be worth his while to undertake the risk if the prospective return is too small.

Therefore, the normal backwardation hypothesis contends that in order to induce the speculators to hold contracts the futures prices should be downward biased estimates of the expected spot prices and are expected to increase over the lives of the contract since the futures prices must equal the spot prices on the maturity date.[2] Recently, Arrow [2] supported the normal backwardation hypothesis, arguing that the futures price at each moment tends to be below the spot price expected on the basis of currently available information and the futures price is revised as new information is available, making the difference between the futures and expected spot prices decrease as the maturity date is getting close. On this basis, he contends that the futures price should be a random process with an upward drift.

On the other hand, the supporters for the normal contango hypothesis (Hardy [11] and Telser [30]) view the futures market as a mechanism where speculators are gambling rather than insuring against changes in prices with risk averse investors who hold positive stocks and thus they are willing to enter the contracts even though the futures price stays above the spot price they expect.

The lack of concensus on the underlying process of the futures prices led a number of researchers subsequently to take a variety of positions for developing models for the futures prices. Working [32] conjectured that there would be no divergencies between the futures prices and the expected spot prices since the futures contracts were motivated only by heterogeneous information even for the risk averse investors. Alchian [1] and Hirshliefer [13] argued that all prices, futures as well as spot prices, would follow martingales. However, this martingale proposition is subject to criticisms of Samuelson [28]. Black [4], Dusak [9], Bodie and Rosanski [3], and Scholes [29] used the convential capital asset pricing model in order to develop pricing equations of the futures contracts. McCulloch [19] showed that there must always be normal backwardation and therefore positive expected profits to speculators in any futures markets. However, this conjecture was due to purely mathematical reasons without any economic reasoning. LeRoy [17] demonstrated that the risk-neutrality is the only necessary and sufficient condition for the futures price to be equal to the expected spot prices.

Motivated by the lack of concensus on the issue of normal backwardation, this paper provides an alternative explanation about the relations between the futures and the expected spot prices.

III. Pricing Model of Futures Contracts

Notations used in this section are as follows:

Xi: commodity X (subscript i represents the specific commodity i)

T: maturity date of futures contracts

Fi(t,T): futures price at time t on commodity i

Pi(τ): spot price of commodity i at time τ

r(τ): continuously compounded interest rate from time τ to τ+1

COV, VAR = σ^2: covariance, variance respectively

Eτ($\tilde{\cdot}$): expected value of the argument ($\cdot$) at time τ

It is well known that the contract price, futures as well as forward, is the value of a claim which will pay a specific number of units of the underlying good on the maturity date (see Cox, Ingersoll and Ross [8], Margrabe [18], Richard and Sundaresan [26], Jarrow and Oldfield [15] and Park [22]). The number in the futures contract is the gross return from "rolling over" strategy in one-period default free bond up to the contract maturity date. Specifically the futures price is the value of a claim that pays $\exp \sum_{\tau=t}^{T-1} r(\tau)$ units of the commodity under consideration at time T.[3] Note that this result is due to the marking-to-market property in the futures contract (see Cox, Ingersoll and Ross [8] for the easy proof).

Once the payoff of a futures contract is known in terms of the number of units that can be paid at time T, a general futures contract pricing model can be developed through the market clearing condition and the maximization of the expected utility function that is assumed the same for all individuals.

Consider the n+1 goods economy composed of Xo, X1, X2.... Xn, where Xo serves as a numeraire good. Investors are assumed to be rational in the sense that, under uncertainty, they are capable of finding every alternatives and choosing the best ones so as to maximize the following lifetime expected utility function that is time additive.

$$E\left[\sum_{\tau=t}^{\infty} \exp(-\alpha\tau)U\{\tilde{X}o(\tau),\ \tilde{X}1(\tau)....\ \tilde{X}n(\tau)\}\right] \tag{1}$$

where α is the utility discount factor that is known at time t and $U(\cdot)$ is a Von Neuman-Morgenstern utility function that is strictly increasing, concave and twice differentiable with respect to Xi: $U\{\tilde{X}o(\tau)\cdot\tilde{X}1(\tau),...\ \tilde{X}n(\tau)\}$ is denoted by $\tilde{U}(\cdot)$ hereafter.[4]

Suppose that two people, a buyer and a seller, commit to β futures contracts, each contract for 1 unit of Xi at the price Fi(t,T) to be paid at time T and the buyer is supposed to pay in terms of Xo. Then, the utility function at time T can be written as

$$U\{\tilde{X}o(T)-\beta\cdot Fi(t,T)\exp\sum_{\tau=t}^{T-1} r(\tau),\tilde{X}1(T),\tilde{X}2(T),....$$

$$\tilde{X}i(T)+\beta\cdot\exp\sum_{\tau=t}^{T-1} r(\tau),....\ \tilde{X}n(T)\} \tag{2}$$

At time t, the market clearing condition and the first order condition for the expected utility maximization in a general equilibrium requires that the differentiation of the expected utility with respect to β conditional $\beta = 0$ is zero. Thus,

$$d\ Et\left[\sum_{\tau=t}^{T-1} \exp(-\alpha\tau)\tilde{U}(T)\right]/d\beta\ \Big|_{\beta=0} = 0 \tag{3}$$

Note that equation (3) is consistent with the no arbitrage condition in that it reflects the essential point of the arbitrage argument that the expected marginal utility from the futures contract should be zero taking no-initial-transfer of money into consideration.[5]

From equations (2) and (3),

$$-Fi(t,T)Et\{\exp(-\alpha\tau)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\tilde{Uo}(T)\}+Et\{\exp(-\alpha\tau)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\tilde{Ui}(T)\} = 0,$$

where Ui and Uo denote the marginal utilities of commodities Xi and Xo respectively. Thus,

$$Fi(t,T) = Et\exp\{\sum_{\tau=t}^{T-1}(r(\tau)-\alpha)\cdot\tilde{Ui}(T)\}/Et\exp\{\sum_{\tau=t}^{T-1}(r(\tau)-\alpha)\cdot\tilde{Uo}(T)\} \tag{4}$$

which is the general futures contract pricing equation.

On the other hand, the relative spot price of commodity Xi in terms of the numeraire good Xo at each time is the ratio of marginal utility of Xi to marginal utility of Xo from the first order condition for utility maximization.

$$Pi(\tau) = Ui(\tau)/Uo(\tau) \tag{5}$$

Thus,

$$Et\{\tilde{Pi}(T)\} = Et\{\tilde{Ui}(T)/\tilde{Uo}(T)\} \tag{6}$$

Note from equations (4) and (6) that the issue whether the futures price follows the "Normal backwardation" process (Keynes [16], Hicks [12], Houthakker [14]) or the "Normal Contango" (Telser [30]) or whether

the futures price is an unbiased estimate of the expected spot price depends on whether the ratio in equation (4) is less or greater than, or equal to $Et\{Ui(T)/Uo(T)\}$.

Using (4), (5) and (6), it is very simple to derive the relation between the futures price and the expected spot price. (See footnote 6 for proof.)

$$Fi(t,T) = Et\,\tilde{Pi}(T) + COV\{\tilde{Pi}(T),\tilde{Uo}(T)\exp\sum_{\tau=1}^{T-1} r(\tau)\}/Et\{\tilde{Uo}(T)\exp\sum_{\tau=1}^{T-1} r(\tau)\} \tag{7}$$

It is obvious now that the relation between the futures price and the expected spot price depends on the covariance between the spot price and the marginal utility of a numeraire good multiplied by $\exp\sum_{\tau=t}^{T-1} r(\tau)$ at the maturity date. The relation between $Fi(t,T)$ and $Et\,\tilde{Pi}(T)$ in (7) is intuitively plausible and consistent with the consumption capital asset pricing model, which states that the risk premium on any asset has the reverse sign as the covariance between the return on the asset and the marginal utility of consumption (see LeRoy [17]). For example, suppose that the random spot price at the maturity date is negatively correlated with the marginal utility of consumption of the numeraire good, holding the effect of $r(\tau)$ fixed for simplicity. In the presence of risk aversion the consumption CAPM contends that a long position in the futures contract will increase consumption risk in terms of the numeraire good and thus the asset will be priced to yield a positive risk premium and in turn the futures price will be less than the expected spot price. It is important to note that the market efficiency concept

does not say anything about the relation between $Fi(t,T)$ and $E_t\tilde{P}i(T)$ in equation (7). The market efficiency is neither sufficient nor necessary for the futures price to be an unbiased estimate of the expected spot price, and thus the normal backwardation or the normal contango is not inconsistent with the market efficiency concept. We can easily derive the relation between the forward price and the expected spot price, which depends on the simple covariance between the spot price and the marginal utility of a numeraire good. However, since the purpose of this paper is to examine the causal relations between the futures price and the expected price, the derivation of the forward contract pricing model corresponding to (7) is suspended to Appendix A.

Testable Hypothesis

In order to derive empirically testable hypotheses, let's assume that the joint distribution of n+1 commodities is multivariate lognormal and that the consumption-investment decision of each individual on the commodities is based on a power utility function such as $(k/(1-\gamma))X^{1-\gamma}$, where the parameter showing the degree of risk aversion is positive. Note that the power utility function is characterized by decreasing absolute risk aversion and constant relative risk aversion.[7]

Since futures contracts as well as forward contracts involve exchanging two commodities at a predetermined price at a future date, they can be analyzed only in two-good models, as pointed out by LeRoy [17].

Then, from the assumption of power utility function,

$$Ut\{\tilde{X}i(T),\tilde{X}o(T)\} = (1/(1-a))\tilde{X}i(T)^{1-a} + (1/(1-b))\tilde{X}o(T)^{1-b} \qquad (8)$$

where positive a and b represent the degree of risk aversion on commodity Xi and Xo respectively.[8]

On the other hand, if Xi is lognormally distributed, then log Xi is normally distributed, and the density of the lognormal distribution is given by

$$F(Xi;\mu,\sigma^2) = (1/Xi\sqrt{2\pi\sigma})\exp\{-1/2\sigma^2(\log Xi-\mu)^2\}$$

where $E(Xi) = \exp(\mu+1/2\cdot\sigma^2)$, and $Var(Xi) = \exp(2\mu+2\sigma^2) - \exp(2\mu+\sigma^2)$

$$E(\log Xi) = \mu \text{ and } Var(\log Xi) = \sigma^2 \qquad (9)$$

Employing the one-period default free discount bond as the numeraire, the explicit relation between the futures price and the expected spot price can be expressed as follows (see Appendix B for the theoretical derivation):[9]

$$\log Fi(t,T) - \log Et\,\tilde{P}i(T) = \sigma o\{COV(I,B) - VAR(B)\}(b-1) \qquad (10)$$

where I and B stand for the logs of marginal utilities of the commodity i and the discount bond respectively.

Examining (10), the following static implications are immediate:

First, if the changes in default-free discount bond price or $r(\tau)$ is nonstochastic ($\sigma o=0$), then futures contract prices are unbiased estimates of expected spot prices.

Second, if the utility function is logarithmic (a=b=1), then the futures price is always an unbiased measure of the expected spot price, regardless of the correlation between the commodity and the one-period

discount bond. This result is consistent with Ohlson [21] arguing that a martingale would be derived even under risk-aversion if utility were logarithmic.

Third, assuming that b > 1, the case of the generalized negative power utility function, the following causal relation can be stated:

$$\frac{COV(I,B)}{VAR(B)} \gtrless 1 \quad \text{implies} \quad Fi(t,T) \gtrless Et\ Pi(T) \tag{11}$$

Equivalently, we can restate the relation in (11) in terms of prices rather than in terms of marginal utilities as follows:

$$\frac{COV(\log S,\ \log V)}{Var(\log V)} \gtrless 1 \quad \text{implies} \quad F(t,T) \gtrless Et\ P(T) \tag{12}$$

where S and V are the prices of the underlying asset and the default-free discount bond respectively.[10] In this paper, the test on the issue of "normal backwardation" is based on the causal relation in (12).

Note however that in a perfect market only unexpected changes in prices would affect the investment opportunities facing investors. Therefore, the covariance and the variance should be interpreted as the covariance of the unexpected changes in commodity prices with the unexpected changes in default-free discount bond prices, and the variance of the unexpected changes in the bond respectively.[11] Then, the ratio of the covariance and the variance in (12) can be tested whether or not it is significantly different from one, by examining the OLS estimate of the slope coefficient of the following regression equation:

$$W = Bo + B_1 Z + \varepsilon \tag{13}$$

where W = log S(T) - log Et S(T)

Z = log V(T) - log Et V(T)

B1 = COV(W,Z)/VAR(Z)

ϵ = disturbance term with zero mean and constant variance

If the models developed in this paper describe the relation between the futures prices and the expected spot price, the futures price should be less (larger) than the expected spot price if the regression coefficient B1 is less (larger) than one. The tests on whether or not B1 in (13) is significantly different from one have been reported in Park and Chen [23] in an attempt to investigate the divergencies between futures and forward prices. The results are reproduced in this paper to further investigate the issue of normal backwardation of futures prices.

One strategy to test the issue of normal backwardation, which is employed in this paper, is to use the returns as follows: if the Normal Backwardation is true underlying process of futures,

$$\frac{EP(T)-F(t,T)}{F(t,T)} > \frac{EP(T)-F(t+1,T)}{F(t+1,T)} > \cdots > \frac{EP(T)-F(T-1,T)}{F(T-1,T)}$$

For example, assuming December 1st, 1982 as the maturity date, using one, two, three, six and twelve months times to maturity,

{EP(Dec/1982)-F(Dec/1981,Dec/1982)}/F(Dec/1981,Dec/1982)

>

{EP(Dec/1982)-F(June/1982,Dec/1982)}/F(June/1982,Dec/1982)

>

{EP(Dec/1982)-F(Sep/1982,Dec/1982)}/F(Sep/1982,Dec/1982)

>

{EP(Dec/1982)-F(Oct/1982,Dec/1982)}/F(Oct/1982,Dec/1982)

>

{EP(Dec/1982)-F(Nov/1982,Dec/1982)}/F(Nov/1982,Dec/1982)

Note that the futures price and the spot price on the maturity date are equal as discussed earlier. In other words, if the normal backwardation is the true process for futures prices, the return defined as {EP(T)-F(t,T)}/F(t,T) should be a function of times to maturity.

More interesting and convincing results concerning the issue of normal backwardation were obtained by running the following regression, taking it into account that the return distribution of futures is a positive function of times to maturity if the normal backwardation is the true underlying process.

$$\text{Return} = \alpha + \beta(\text{Time}) \tag{14}$$

where Time = 1 if the return is for one month

Time = 2 if the return is for two months

Time = 3 if the return is for three months

Time = 6 is the return is for six months

Time = 12 if the return is for twelve months

If the normal backwardation is the true process and thus the return is a positive function of times to maturity, β should be positive significantly. So, by examining the coefficient of the regression equation (14), we can test the null hypothesis:

$$H0: \quad \beta = 0$$

IV. Data and Results

A. Data

The data in this paper include the spot prices and the closing prices of futures contracts on six basic commodities and four foreign currencies, Gold, Silver, Silver Coin, Platinum, Copper, Plywood, Swiss Franc, German Mark, British Pound and Japanese Yen, which are observed on the first trading dates of each month during July, 1977 and December, 1981. Five different times to maturity futures prices of each commodity are used: one, two, three, six and twelve months times to maturity futures prices.[12] All of these observations were obtained from the Wall Street Journal and the Journal of Commerce.

Additionally, one, two, three, six and twelve months Treasury-Bill prices were used as the default-free discount bond prices for the corresponding terms. All of these data were obtained from An Analytical Record of Yields and Yield Spreads, published by Solomon Brothers.

B. Results

The tests on whether B1 defined in (13) is significantly different from one have been reported in Park and Chen [23], using the forward prices with one, two, three and six months maturities as proxies for the expected prices for the corresponding time periods: the futures prices with twelve months maturities were used as the proxies for the corresponding expected prices since the T-bill forward contract with one year times to maturity was not available in the American Board of Trade in New York, which was the only organized exchange for forward contracts. The results are reproduced and summarized here in order to investigate the issue of normal backwardation, which is the primary purpose of this paper.

Table 1 summarizes the results of the test whether B1 defined in equation (13) is greater or less than or equal to one. The regression coefficients of all physical commodities, Gold, Silver, Silver Coin, Platinum, Copper and Plywood in all maturities, one, two, three, six and twelve months, turned out to be significantly less than one at 5% level except the twelve months maturity Silver Coin and Plywood.[13]

In contrast to the results on commodities, all of the coefficients for the foreign currencies such as Swiss Franc, German Mark, British Pound and Japanese Yen turned out to be insignificantly different from one at 5% level, except one-month German Mark.

Table 2 provides the results of tests on whether the returns on futures contracts are significantly different from zero, using ex-post realized spot prices as proxies for their ex-ante expectations. Note that the model developed earlier conjects that if B1 in (13) is less (equal to) than one, then the returns on futures contracts are positive (equal to zero) and β coefficient in (14) is greater (equal to) than zero.

Except one commodity, Plywood, the return distributions of futures contracts strongly tend to be consistent with the basic hypothesis that they depend on the covariances between the commodities and the discount bonds (T-bill in this paper). Thirty-one commodities out of forty-nine (63.3%) are consistent with the hypothesis; if plywood is excluded, thirty-one out of forty-four commodities (70.5%) are consistent at 5% significance level. These results support the model and its implication developed earlier.

More interesting and convincing results of the test on normal backwardation are presented in Table 3. As expected in the light of the result of Tables 1 and 2, the beta estimations of Gold, Silver, Silver Coin,

TABLE 1

Tests on the Ratio of Covariance to Variance

$$W = B_0 + B_1 Z + \varepsilon \quad [a]$$

$$H0: \; B_1 = 1$$

Mat(T)	obs	B0	T0	B1	T1 [d]	H0
Gold						
1	19	0.1616	2.06	-1.5218	-4.0118	< [b]
		(0.0783)		(0.6286)	[c]	
2	18	0.2368	1.99	-2.6525	-3.4107	<
		(0.1191)		(1.0709)		
3	17	0.2145	1.81	-3.2562	-3.4357	<
		(0.1186)		(1.2388)		
6	14	-0.0410	-0.49	-4.1391	-3.1839	<
		(0.0830)		(.1.6141)		
12	30	0.2728	1.77	-3.4984	-2.3879	<
		(0.1541)		(1.8838)		
Silver						
1	17	0.1798	1.40	-1.6558	-2.4807	<
		(0.1285)		(0.0706)		
2	16	0.3904	2.25	-4.0998	-3.2259	<
		(0.1736)		(1.5809)		
3	15	0.5148	2.79	-6.5602	-3.9372	<
		(0.1847)		(1.9202)		
6	12	0.1115	0.66	-8.9489	-3.1197	<
		(0.1691)		(3.1891)		
12	30	0.2239	0.90	-8.8865	-3.0334	<
		(0.2484)		(3.2592)		
Silver Coin						
1	19	0.1791	1.92	-1.6961	-3.5919	<
		(0.0935)		(0.7506)		
2	18	0.3121	2.53	-3.4031	-3.9711	<
		(0.1233)		(1.1088)		
3	17	0.2879	2.06	-4.1312	-3.5157	<
		(0.1397)		(1.4595)		
6	14	-0.0106	-0.09	-5.0868	-2.0929	<
		(0.1163)		(2.2603)		
12	30	0.2564	1.15	3.1550	0.3830	=
		(0.2233)		(5.6260)		

Platinum						
1	17	0.1115	1.28	-1.0769	-2.7480	<
		(0.0907)		(0.7558)		
2	16	0.1873	1.43	-2.1473	-2.6339	<
		(0.1312)		(1.1949)		
3	15	0.2193	1.59	-3.3861	-3.0625	<
		(0.1378)		(1.4322)		
6	12	-0.0677	-0.61	-3.8218	-2.3070	<
		(0.1108)		(2.0901)		
12	10	-0.6171	-3.66	-4.1615	-2.1880	<
		(0.1684)		(2.3590)		
Copper						
1	19	0.0722	1.46	-0.7565	-4.4267	<
		(0.0494)		(0.3968)		
2	18	0.0951	1.98	-1.2664	-5.2427	<
		(0.0481)		(0.4323)		
3	17	0.1338	1.74	-2.0027	-3.7283	<
		(0.0771)		(0.8054)		
6	14	-0.0206	-0.59	-2.1933	-4.6981	<
		(0.0350)		(0.6797)		
12	10	-0.3277	-3.97	-2.0470	-4.7402	<
		(0.0824)		(0.6428)		
Plywood						
1	19	0.1331	2.92	-1.1924	-5.9853	<
		(0.0456)		(0.3663)		
2	18	0.1858	3.24	-1.9692	-5.7532	<
		(0.0456)		(0.3663)		
3	17	0.2131	3.33	-2.7392	-5.5951	<
		(0.0640)		(0.6683)		
6	14	0.1190	2.65	-4.2647	-6.0271	<
		(0.0449)		(0.8735)		
12	10	-0.1487	-1.59	-1.2634	-1.2104	=
		(0.0938)		(1.8699)		
S.F. (Swiss Franc)						
1	19	-0.0308	-0.68	0.3376	-1.8565	=
		(0.0445)		(0.3568)		
2	18	0.0186	0.24	0.0337	-1.4057	=
		(0.0764)		(0.6874)		
3	17	0.0533	0.60	-0.1312	-1.2118	=
		(0.0894)		(0.9335)		
6	31	-0.0517	-0.91	1.5281	0.439	=
		(0.0568)		(1.2016)		
12	10	0.1244	5.01	-0.2825	-0.8579	=
		(0.0248)		(1.4590)		

G.M. (German Mark)						
1	19	-0.0253	-0.70	0.3326	-2.2903	<
		(0.0360)		(0.2914)		
2	18	0.0042	0.07	0.2576	-1.3236	=
		(0.0624)		(0.5609)		
3	17	0.0428	0.59	0.1147	-1.1672	=
		(0.0726)		(0.7585)		
6	31	-0.0351	-0.84	1.5575	0.6275	=
		(0.0417)		(0.8884)		
12	10	0.1720	5.45	1.8076	0.9731	=
		(0.0316)		(0.8299)		
B.P. (British Pound)						
1	19	-0.0877	-2.28	0.7749	-0.7301	=
		(0.0384)		(0.3083)		
2	18	-0.1069	-1.90	1.1730	0.3424	=
		(0.0562)		(0.5053)		
3	17	-0.1250	-2.04	1.7772	1.2147	=
		(0.0613)		(0.6398)		
6	30	-0.0524	-1.12	0.6208	-0.3881	=
		(0.0467)		(0.9770)		
12	10	0.1014	1.17	-0.1976	-0.6910	=
		(0.0869)		(1.7331)		
J.Y. (Japanese Yen)						
1	19	-0.0426	-0.94	0.3640	-1.7530	=
		(0.0452)		(0.3628)		
2	18	-0.0555	-0.97	0.5846	-0.8107	=
		(0.0569)		(0.5124)		
3	17	-0.1030	-1.99	1.2802	0.5179	=
		(0.0518)		(0.5410)		
6	27	0.0055	0.10	0.1477	-0.7182	=
		(0.0564)		(1.1868)		
12	10	-0.0260	-0.40	-0.6219	-1.2369	=
		(0.0658)		(1.3113)		

a - As defined in equation (13).

b - The symbol '<' represents that B1 is significantly less than 1.

The symbol '=' represents that B1 is not significantly different from 1.

The symbol '>' represents that B1 is significantly greater than 1.

c - The figures in brackets show standard errors of estimations.

d - 't' statistics of B0 and B1 are against 0 and 1 respectively.

'Mat(T)' is the times to maturity T (months).

'obs' shows the number of observations.

TABLE 2

Tests on Futures Return

Futures return; (Spot-F)/F x 100,
where F stands for futures prices

	Mat	1	2	3	6	12	
Gold	Mean	1.7183	2.6673	7.3091	11.2985	34.0844	
	S.D.	9.9766	15.1810	19.9583	32.7156	54.1600	
	obs	48	36	47	47	41	
	T	1.19	1.05	2.51	2.37	4.03	a
	Ho	=	=	>	>	>	b
Silver	Mean	1.5021	4.6289	8.6289	26.3712	52.0149	
	S.D.	16.0581	30.1168	39.2153	79.8181	121.1796	
	obs	48	43	47	44	39	
	T	0.65	1.01	1.51	2.19	2.68	
	Ho	=	=	=	>	>	
Silver Coin	Mean	10.1353	26.1386	14.2014	40.8408	88.6825	
	S.D.	35.7442	35.7098	25.7421	75.6812	115.8682	
	obs	13	12	24	22	20	
	T	1.02	2.54	2.70	2.53	3.42	
	Ho	=	>	>	>	>	
Platinum	Mean	-2.7427	2.7497	8.7860	14.5346	33.4428	
	S.D.	10.4907	21.3144	20.3147	32.5748	43.3807	
	obs	19	14	31	28	24	
	T	-1.14	0.48	2.41	2.36	3.78	
	Ho	=	=	>	>	>	
Copper	Mean	2.3144	0.8937	2.2677	6.8264	10.035[illegible]	
	S.D.	9.7423	13.9808	15.1547	20.1526	28.423[illegible]	
	obs	46	37	43	42	37	
	T	1.61	0.39	0.98	2.20	2.15	
	Ho	=	=	=	>	>	
Plywood	Mean	-2.3121	-2.8359	-4.9918	-5.5802	-8.2999	
	S.D.	5.8155	8.4108	8.9507	9.7088	9.697[illegible]	
	obs	11	6	24	20	21	
	T	-1.32	-0.83	-2.73	-2.57	-3.92	
	Ho	=	=	<	<	<	

S.F.	Mean	-0.0071	-2.3225	0.2193	-0.5714	12.2776
	S.D.	4.8733	7.4841	9.1654	13.2367	12.2776
	obs	23	17	24	31	20
	T	-0.01	-1.28	0.12	-0.24	-0.30
	Ho	=	=	=	=	=
G.M.	Mean	-0.4414	-1.5567	-1.0509	-2.6595	-2.1869
	S.D.	3.3313	3.9651	6.1288	9.3656	11.1761
	obs	23	15	25	31	17
	T	-0.64	-1.52	-0.86	-1.58	-0.81
	Ho	=	=	=	=	=
B.P.	Mean	0.2614	-0.3442	1.2592	2.9467	6.5500
	S.D.	2.8422	3.5062	6.2819	9.2760	12.5847
	obs	19	13	23	30	15
	T	0.40	-0.35	0.96	1.74	2.02
	Ho	=	=	=	=	=
J.Y.	Mean	-0.3479	-1.5395	-0.3767	-0.4777	c
	S.D.	4.4865	6.2192	8.7806	12.0836	
	obs	23	17	23	27	
	T	-0.37	-1.02	-0.21	-0.21	
	Ho	=	=	=	=	

a Computed against 0.

b The symbol '<' represents that the mean is significantly less than 0.

The symbol '=' represents that the mean is not significantly different from 0.

The symbol '>' represents that the mean is significantly greater than 0.

c No observation is available in twelve months times to maturity Japanese Yen futures prices.

TABLE 3

Normal Backwardation of Futures Prices

Futures Return = $\alpha + \beta$ Time.

	α	to	β	t1 b	Ho c	R^2
Gold (219) a	-3.2525 (3.2139)	-1.01	2.9646 (0.5249)	5.65	>	0.1282
Silver (221)	-4.5894 (6.8893)	-0.67	4.6275 (1.1565)	4.00	>	0.0681
Silver Coin (91)	-0.2366 (12.1992)	-0.02	7.2348 (1.8519)	3.91	>	0.1464
Platinum (116)	-6.1696 (4.3771)	-1.41	3.2580 (0.6792)	4.80	>	0.1680
Copper (205)	0.6923 (1.9699)	0.35	0.6852 (0.3274)	2.09	>	0.0155
Plywood (82)	-2.7879 (1.6864)	-1.65	-0.4701 (0.2416)	-1.95	=	0.0452
S.F. (115)	1.0262 (1.5443)	0.66	-0.4524 (0.2525)	1.79	=	0.0407
G.M. (111)	-0.8486 (1.1523)	-0.74	-0.1645 (0.1951)	-0.84	=	0.0065
B.P. (100)	-0.7025 (1.2890)	-0.54	0.4041 (0.2173)	1.86	=	0.0731
J.Y. (92)	1.8498 (1.6916)	1.09	-0.8012 (0.4111)	-1.49	=	0.0507

a The figure in the bracket of this column is the number of observations.

b Computed against 0.

c The symbol '<' represents that β is significantly less than 0.

The symbol '=' represents that β is not significantly different from 0.

The symbol '>' represents that β is significantly greater than 0.

Platinum and Copper, are significantly greater than zero at 5% level, and thus show that the returns on the futures contracts are positive functions of times to maturity and in turn support the normal backwardation process. Only exception is Plywood again among physical commodities.

On the other hand, the beta coefficients of all foreign exchange rates, Swiss Franc, German Mark, British Pound and Japanese Yen are not significantly different from zero, implying that the futures contract prices of these foreign exchanges follow neither the normal backwardation nor the normal contango process. These results are once again consistent with the basic hypothesis and provide strong supports for the models of futures contracts developed earlier.

V. Conclusion

This paper has developed the theoretical models of futures contracts through the market clearing condition and the first-order condition for the expected utility maximization, in the absence of transaction costs and in the presence of uncertainty. Based upon the models, the causal relations between the futures prices and the expected spot prices were derived. Specifically, we examined under what conditions the systematically downward (normal backwardation) or upward (normal contango) biased futures prices against the expected spot prices might be observable. The derived implications of the theoretical models are significantly consistent with the empirical findings using ten commodities (six physical commodities and four foreign currencies) with five different times to maturity each. In the process, the conventional hypothesis could be

rejected that the systematic bias of futures prices against expected spot prices might be due to market inefficiency: the systematic bias, if any, is not necessarily inconsistent with market efficiency. Nevertheless, since the models are formulated in a simplified economy, the analysis in this paper can never be perfect in explaining the sources of deviations from the models. However, the purpose of this paper is not so much to introduce all the many factors that could theoretically influence futures prices simultaneously into one equation as it is to find the best explanation for the systematic relations between the futures price and the expected spot price in the simple economy.

Footnotes

[1]See LeRoy [17] for a survey of theory on expectation models of asset prices.

[2]See Black [4], Cox, Ingersoll and Ross [8], Richard and Sundaresan [26], Jarrow and Oldfield [15] and Park [22] for the equivalence of futures and spot prices on the maturity date.

[3]The number in the forward contract is the gross return from "going long" strategy in a default free discount bond maturing at the same time as the forward contract. For descriptions of fundamental differences between futures and forward contracts, see the references in footnote 2.

[4]Throughout this paper, the consideration of margin requirement is ruled out; it is important to note that margin requirements are not partial equity payments against the market value of the commodity represented by the contract, as it is when buying common stocks, but a guarantee in the event of adverse price movements.

Nevertheless, a controversy over margin requirements exists in connection with investors' optimal portfolio construction. For example, Telser [30] argues that margin requirement should be incorporated in the pricing equations because they may disrupt individual investor's optimal portfolio allocation and thus induce costs even though they are in the form of interest-earning securities like Treasury bills. However, as long as individual investors can borrow in a perfect capital market against their portfolios to buy Treasury bills for the purpose of posting them as margin requirements, it doesn't induce any cost. In other words, the opportunity cost for posting margin requirements is zero, assuming no transaction costs in a perfect capital market.

[5]A market clearing condition simply requires that demand equals supply for each commodity, in which the market is cleared of all outstanding units of assets or contracts. Thus, even though consumers face different consumption investment opportunity sets and thus have heterogeneous perceptions on the probability distributions of risk-return on each asset or claim, it poses no problem for the determination of a market equilibrium.

[6]From equation (4),

$$Fi(t,T) = Et\{\tilde{Ui}(T)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\}/Et\{\tilde{Uo}(T)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\}$$

since α is deterministic

$$= Et\{\tilde{Pi}(T)\cdot\tilde{Uo}(T)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\}/Et\{\tilde{Uo}(T)\exp\sum_{\tau=t}^{T-1} r(\tau)\}$$

from (5)

$$= COV\{\tilde{Pi}(T), \tilde{Uo}(T)\exp\sum_{\tau=t}^{T-1} r(\tau)\} + Et\,\tilde{Pi}(T)\cdot Et\,\tilde{Uo}(T)\exp\sum_{\tau=t}^{T-1} r(\tau)$$
$$/Et\{\tilde{Uo}(T)\exp\sum_{\tau=t}^{T-1} r(\tau)\}$$

$$= Et\,\tilde{Pi}(T) + COV\{\tilde{Pi}(T),\tilde{Uo}(T)\exp\sum_{\tau=t}^{T-1} r(\tau)\}/Et\{\tilde{Uo}(T)\exp\sum_{\tau=t}^{T-1} r(\tau)\}$$

which is the equation (7). This equation (7) is quite similar to the models of Richard and Sundaresan [15] obtained from a quite different approach.

[7] If $U(X) = (k/1-\gamma)X^{1-\gamma}$, $U^1 = X^{-\gamma}$, $U^{11} = -k/\gamma\ X^{-\gamma-1}$
Thus, the absolute risk aversion,

$$R_A(X) = -U^{11}/U^1 = -k/\gamma\ X^{-\gamma-1}/X^{-\gamma} = k/\gamma\ X^{-1}$$

$$R_A{}^1(X) = -k/\gamma\ X^2 < 0,$$

which implies the decreasing absolute risk aversion.

Also, the relative risk aversion,

$$R_R(X) = R_A(X)X = k/\gamma\ X^{-1}\ X = k/\gamma$$

$$R_R{}^1(X) = 0,$$

which implies the constant relative risk aversion.

[8] The general power utility function is given by $(Ki/1-\gamma)Xi^{1-\gamma}$. In this paper, Ki is assumed to equal one for simplicity with no loss of generality for comparison between futures prices and expected spot prices.

[9] There is no economic reason that the discount bond should be chosen necessarily. However, the choice of a discount bond is theoretic convenient to provide a rigorous foundation for the test methods. Suppos that there is a futures contract in every state of the world. In the other words, a futures contract is an asset with a payoff in the next period that is equal to the state price that is uncertain, assuming the existence of a futures contract on each state that expires at every instant and anot created that matures in the next instant. Then it is well kn that a set of futures contract plus a risk free one-period discount in achieve the complete market in Arrow and Debrew sense.

Thus, the choice of the one-period discount bond is believed to be a reasonable choice as a numeraire good.

[10]It is well known that any specific choice of utility function (for example, power utility in this paper) implicitly specifies a conditional distribution assumption for consumption if marginal utility is conditionally joint lognormal with prices. Additionally, investors under the generally accepted assumptions in this paper will behave so as to equate their marginal rates of substitution for consumption at each time with price ratios of traded financial claims (the underlying commodity and the discount bond in this paper) which guarantee consumption payouts in the future. Therefore, the switch from (11) to (12) is not so restrictive as it seems in the economy assumed in this paper.

[11]The tests based on the simple covariance between the commodity price and the default-free discount bond price, and the variance of the bond price, have been done. However, they don't deviate significantly from those reported here. They are available from the author upon request.

[12]In fact, unlike forward contracts, futures contracts are standardized by specific maturity dates instead of specific times to maturity. But in the spirit of Park and Chen [23], this paper assumes specific times to maturity without the loss of generality. The forward prices in the organized exchange (the American Board of Trade in New York) are available only from July 1977. Thus, the data in Park and Chen and this paper was restricted to the data since then.

[13]The coefficient B1 of twelve-month Platinum is less than and equal to one at 10% and 5% significance levels respectively.

References

1. Alchian, Armen, "Information, Martingales and Prices," Swedish Journal of Economics, 76, March 1974.

2. Arrow, Kenneth J., "Futures Markets: Some Theoretical Perspectives," Journal of Futures Markets, Vol. 1, Number 2, 1981.

3. Bodie and Rosansky, "Risk and Return in Commodity Futures," Working Paper of CSFM, Columbia Business School, 1979.

4. Black, F., "The Pricing of Commodity Contracts," Journal of Financial Economics, January, 1976.

5. Burger, Lang and Rasche, "The Treasury Bill Futures Markets and Market Expectations of Interest Rates," Federal Reserve Bank of St. Louis, Vol. 59, 1977.

6. Cootner, P., "Returns to Speculators: Telser vs Keynes," Journal of Political Economy, 68, August 1960.

7. Cornell, Bradford, "Spot Rates, Forward Rates and Exchange Market Efficiency," Journal of Financial Economics, 5, August 1977.

8. Cox, Ingersoll and Ross, "The Relation Between Forward Prices and Futures Prices," Journal of Financial Economics, December, 1981.

9. Dusak, "Futures Trading and Investor Return: An Investigation of Commodity Markets Risk Premiums," Journal of Political Economy, October, 1973.

10. Geweke, J. and Feige, E., "Some Joint Tests of the Efficiency of Markets for Forward Foreign Exchange," Review of Economics and Statistics, 61, August 1979.

11. Hardy, Charles, "Risk and Risk Bearing," University of Chicago Press, 1940.

12. Hicks, "Value and Capital: An Inquiry into Some Fundamental Principles of Economic Theory," Second Edition, Oxford Press, 1939.

13. Hirshleifer, T., "Speculation and Equilibrium: Information, Risk and Markets," Quarterly Journal of Economics, 89, November 1975.

14. Houthakker, H. S., "Can Speculators Forecast Prices," Review of Economics and Statistics, May, 1957.

15. Jarrow and Oldfield, "Forward Contracts and Futures Contracts," Journal of Financial Economics, December, 1981.

16. Keynes, J. M., "A Treatise on Money," Harcourt, Brace and Company, New York, 1930.

17. LeRoy, S. F., "Expectations Models of Asset Prices: A Survey of Theory," Journal of Finance, March 1982.

18. Margrabe, W., "A Theory of Forward and Futures Prices," Working Paper, The Wharton School, University of Pennsylvania, 1978.

19. McCulloch, J. H., "Comment on 'Risk, Interest and Forward Exchange'," Quarterly Journal of Economics, 1975.

20. O'Brien and Schwarz, "Ex Ante Evidence of Backwardation/Contango in Commodities Futures Market," The Journal of Futures Markets, Vol. 2, No. 2, 1982.

21. Ohlson, James, "Risk Aversion and the Martingale Property of Stock Prices: Comment," International Economic Review, 18, February 1977.

22. Park, H. Y., "A Theoretical and Empirical Investigation of Forward and Futures Prices," unpublished dissertation thesis, the Ohio State University, August, 1982.

23. Park, H. and Chen, A., "Divergencies Between Futures and Forward Prices: An Empirical Investigation of the Marking-To-Market Effects," Working Paper No. 928, The University of Illinois at Urbana-Champaign, 1983.

24. Poole, W., "Using T-bill Futures to Gauge Interest Rate Expectations," Federal Reserve Bank of San Francisco, Spring 1978.

25. Rendleman, R. and Carabini, C., "The Efficiency of the Treasury Bill Futures Markets," Journal of Finance, September 1979.

26. Richard, S. F. and Sundaresan, M., "A Continuous Time Equilibrium Model of Forward Prices and Futures Prices in a Multigood Economy," Journal of Financial Economics, December, 1981.

27. Rockwell, C., "Normal Backwardation, Forecasting and the Returns to Commodity Futures Traders," Food Research Institute Studies, 7, 1969.

28. Samuelson, D. A., "Proof that Properly Anticipated Prices Fluctuate Randomly," Industrial Management Review, Spring, 1965.

29. Scholes, M., "The Economics of Hedging and Spreading in Futures Markets," Journal of Futures Markets, 1, Summer 1981.

30. Telser, Lester G., "Futures Trading and the Storage of Cotton and Wheat," Journal of Political Economy, June, 1958.

31. Wilson, J., "Comment on 'Ex Ante Evidence of Backwardation/Contango in Commodities Futures Markets," The Journal of Futures Markets, Vol. 2, No. 2, 1982.

32. Working, M., "Futures Trading and Hedging," American Economic Review, 43, June 1953.

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APPENDIX A

On the same line of logic, forward contract pricing models can be formulated. As noted in footnote 3, the forward price is the value of a claim that pays $B(t,T)^{-1}$ units of the commodity under consideration at its maturity date T, where B(t,T) is the price as of time t of a default free bond paying one dollar at time T. (See the references quoted in footnote 2.)

Like the futures contract, if a person engages in β units of forward contracts, each unit of the contract for one unit of Xi at the price fi(t,T) by promising to pay for it with Xo, his utility function at time T can be written as

$$U\{\tilde{X}o(T) - \beta \cdot fi(t,T)B(t,T)^{-1}, \tilde{X}1(T), \ldots. \tilde{X}i(T) + \beta \cdot B(t,T)^{-1}, \ldots. \tilde{X}n(T)\}$$

From equation (3),

$$-fi(t,T)B(t,T)^{-1}Et[\exp(-(T-t)\alpha)\tilde{U}o(T)] + B(t,T)^{-1}Et[\exp(-(T-t)\alpha)\tilde{U}i(T)] = 0$$

$$fi(t,T) = B(t,T)^{-1}Et[(\exp(-(T-t)\alpha)\tilde{U}i(T)]/B(t,T)^{-1}Et[(\exp(-(T-t)\alpha)\tilde{U}o(T)]$$

$$fi(t,T) = Et\tilde{U}i(T)/Et\tilde{U}o(T)$$

which is the general forward price of commodity i from the market clearing condition and the first order condition for the expected utility maximization.

Thus,

$$fi(t,T) = Et\tilde{U}i(T)/Et\tilde{U}o(T)$$

$$= Et\{\tilde{U}o(T) \cdot \tilde{P}i(T)\}/Et\tilde{U}o(T) \qquad \text{From equation (5)}$$

$$= [COVt\{\tilde{U}o(T), \tilde{P}i(T)\} + Et\tilde{U}o(T) \cdot Et\tilde{P}i(T)]/Et\tilde{U}o(T)$$

$$= Et\tilde{P}i(T) + COVt\{\tilde{U}o(T), \tilde{P}i(T)\}/Et\tilde{U}o(T)$$

APPENDIX B

Notations used in this appendix are as follows:

$\sigma^2 i(\tau)$: variance of $\log \tilde{X}i(\tau)$ at time τ.

$\sigma^2 o(\tau)$: variance of $\log \tilde{X}o(\tau)$ at time τ.

$\sigma^2 r(\tau)$: variance of $\sum_{\tau=t}^{T-1} r(\tau)$.

ρio: correlation between $\log \tilde{X}i(\tau)$ and $\log \tilde{X}o(\tau)$.

ρir: correlation between $\log \tilde{U}i(\tau)$ and $\sum_{\tau=t}^{T-1} r(\tau)$.

ρor: correlation between $\log \tilde{U}o(\tau)$ and $\sum_{\tau=t}^{T-1} r(\tau)$.

$$\tilde{\phi}(\tau) = \log \tilde{U}i(\tau)$$

$$\tilde{\Phi}(\tau) = \log \tilde{U}o(\tau)$$

$$\tilde{\phi}^1(\tau) = \log \tilde{U}i(\tau)\exp \sum_{\tau=t}^{T-1} r(\tau) = \log \tilde{U}i(\tau) + \sum_{\tau=t}^{T-1} r(\tau)$$

$$\tilde{\Phi}^1(\tau) = \log \tilde{U}o(\tau)\exp \sum_{\tau=t}^{T-1} r(\tau) = \log \tilde{U}o(\tau) + \sum_{\tau=t}^{T-1} r(\tau)$$

$$\tilde{Z}(\tau) = \tilde{\phi}(\tau) - \tilde{\Phi}(\tau) = \tilde{\phi}^1(\tau) - \tilde{\Phi}^1(\tau).$$

From equation (8),

$$\tilde{U}i(T) = \tilde{X}i(T)^{-a},\ \tilde{U}o(T) = \tilde{X}o(T)^{-b}$$

$$\tilde{\phi}(T) = \log \tilde{U}i(T) = -a \log \tilde{X}i(T)$$

$$\tilde{\Phi}(T) = \log \tilde{U}o(T) = -b \log \tilde{X}o(T)$$

$$\tilde{Z}(T) = \tilde{\phi}(T) - \tilde{\Phi}(T) = \log\{\tilde{U}i(T)/\tilde{U}o(T)\} = \log \tilde{P}i(T)$$

$$Et\tilde{Z}(T) = Et\tilde{\phi}(T) - Et\tilde{\Phi}(T)$$

$$\text{Var}\ \tilde{\phi}(T) = \text{Var}\{\log \tilde{U}i(T)\} = a^2 \cdot \sigma^2 i(T)$$

$$\text{Var}\ \tilde{\Phi}(T) = \text{Var}\{\log \tilde{U}o(T)\} = b^2 \cdot \sigma^2 o(T)$$

$$\text{Var}\,\tilde{\phi}^{1}(T) = \text{Var}\{\log \tilde{U}i(T) + \sum_{\tau=t}^{T-1} r(\tau)\}$$

$$= a^2 \cdot \sigma^2 i(T) + \sigma^2 r(T) + 2\rho ir\ a\ \sigma i(T)\sigma r(T)$$

$$\text{Var}\,\tilde{\Phi}^{1}(T) = \text{Var}\{\log \tilde{U}o(T) + \sum_{\tau=t}^{T-1} r(T)\}$$

$$= b^2 \cdot \sigma^2 o(T) + \sigma^2 r(T) + 2\rho or\ b\sigma o(T)\sigma r(T)$$

From (4) and (9),

$$Fi(t,T) = Et\{\tilde{U}i(T) \cdot \exp \sum_{\tau=t}^{T-1} r(\tau)\} / Et\{\tilde{U}o(T) \exp \sum_{\tau=t}^{T-1} r(\tau)\}$$

$$= \exp\{E\tilde{\phi}^{1}(T) + 1/2(a^2 \cdot \sigma^2 i(T) + \sigma^2 r(T) + 2\rho ir\ a\sigma i(T)\sigma r(T))\} /$$

$$\exp\{E\tilde{\Phi}^{1}(T) + 1/2(b^2\sigma^2 o(T) + \sigma^2 r(T) + 2\rho or\ b\sigma o(T)\sigma r(T))\}$$

$$= \exp\{Et\tilde{Z}(T) + 1/2((a^2\sigma^2 i(T) + 2\sigma r(\rho ir\ a\sigma i(T) - \rho or\ b\sigma o(T))$$

$$-b^2\sigma^2 o(T))\} \qquad (B\text{-}1)$$

Also,

$$Et\tilde{P}i(T) = Et\{\tilde{U}i(T)/\tilde{U}o(T)\} = Et\{\exp\tilde{\phi}(T)/\exp\tilde{\Phi}(T)\} = Et\{\exp(\tilde{\phi}(T) - \tilde{\Phi}(T)$$

$$\text{Var}\{\log \tilde{P}i(T)\} = \text{Var}\{\tilde{\phi}(T) - \tilde{\Phi}(T)\}$$

$$= a^2 \cdot \sigma^2 i(T) + b^2 \cdot \sigma^2 o(T) - 2\rho io\ ab\sigma i(T)\sigma o(T)$$

Thus, from (9)

$$Et\tilde{P}i(T) = \exp\left[Et\tilde{Z}(T) + 1/2\{a^2 \cdot \sigma^2 i(T) + b^2 \cdot \sigma^2 o(T) - 2\rho io\ ab\sigma i(T)\sigma o(T)\}\right] \qquad (B\text{-}2)$$

Equations (B-1) and (B-2) show the explicit form pricing models for futures contract and the expected future spot price under the given assumptions.

Subtracting (B-2) from (B-1) after taking logarithm

$$\begin{aligned}\log Fi(t,T) - \log EtPi(T) &= -b^2 \cdot \sigma^2 o(T) + \sigma r\{\rho ir\ a\sigma i(T) \\ &\quad -\rho or\ b\sigma o(T)\} + \rho io\ ab\sigma i(T)\sigma o(T) \\ &= \sigma o\{a\ \rho io\ \sigma i(T) - b\sigma o(T)\}(b-1) \\ &= \sigma o\{a \cdot b\ \rho io\ \sigma i(T)\sigma o(T) - b^2\sigma\hat{o}(T)\}(b-1)\end{aligned}$$

(B-3)

assuming the one-period default free discount bond as a numeraire good. (B-3) is equivalent to equation (10).

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